Midterm will be after break.

- Either in class or evening.

<u>Remember:</u> Let A be a set.

- A <u>relation</u> on A is a subset $R \subseteq A \times A$

For elements $a, b \in A$, we say aRb "a related to b"

if $(a,b) \in R$.

<u>Ex:</u> Suppose $A = \{1,2,3\}$. We want to describe the relation "$a \equiv b \pmod 2$"

	1	2	3
1	✓	✗	✓
2	✗	✓	✗
3	✓	✗	✓

So $R = \{(1,1), (2,2), (3,3), (1,3), (3,1)\}$.

This example is an $\boxed{\text{equivalence relation}}$ because.

1. Reflexive: $\forall a \in A, (aRa)$ not conditional!
2. Symmetric: $\forall a,b \in A, (aRb \Rightarrow bRa)$ Conditional.
3. Transitive: $\forall a,b,c \in A, ((aRb \wedge bRc) \Rightarrow aRc)$ conditional.

Example: $A = \{1,2,3\}$. Suppose $R = \{\}$

- Is R reflexive? This is the same as asking "Does R contain $(1,1)$, $(2,2)$, and $(3,3)$?" NO!

- Is R symmetric? Yes, because for every a,b aRb is false. So $(aRb \Rightarrow bRa)$ is true.

- Is R transitive? Yes, for the same reason.

Proving a relation is an equivalence relation

Prop: Let $A = \mathbb{Z}$. Fix any integer $n \geq 1$. Then the relation $R = \{(a,b) : a \equiv b \pmod{n}\}$ is an equivalence relation.

Proof: Reflexive: I need to check "$\forall a \in \mathbb{Z}, \quad a \equiv a \pmod{n}$"

~~For every~~ This is equivalent to checking

$n \mid a-a$

$n \mid 0$ but this is true because

$0 = n \cdot 0$.

Symmetric: "$\forall a,b \in \mathbb{Z}, (a \equiv b \pmod{n} \Rightarrow b \equiv a \pmod{n})$"

Suppose $a \equiv b \pmod{n}$

Then $n \mid a-b$ (Definition)

Then there is some $x \in \mathbb{Z}$, $a-b = n \cdot x$

$b-a = -n \cdot x$

So $n \mid b-a$. So $b \equiv a \pmod{n}$

Transitive: "$\forall a,b,c \in \mathbb{Z}, ((a \equiv b \pmod{n} \wedge b \equiv c \pmod{n}) \Rightarrow a \equiv c \pmod{n})$"

Suppose $a \equiv b \pmod{n}$ and $b \equiv c \pmod{n}$

$n \mid a-b$ $\qquad$ $n \mid b-c$

$\exists x \quad a-b = nx$, $\qquad \exists y \quad b-c = ny$

$(a-c) = (a-b) + (b-c) = nx + ny = n(x+y)$

So $n \mid a-c$, so $a \equiv c \pmod{n}$. ∎

Prove each is an equivalence relation

Exercise 1: $A = \mathbb{R}$, $R = \{(a,b) : a-b \in \mathbb{Z}\}$.

Exercise 2: $A = \mathbb{Z}$, $R = \{(a,b) : 3a+5b \text{ is even}\}$.

Exercise 3: A = the set of all lines in the Cartesian plane

$R = \{(L_1, L_2) :$ either L_1 and L_2 are the same line, or they are parallel. $\}$.

Proof of 3: Reflexive: "For every line L, L is either equal to or parallel to L."

L is equal to itself.

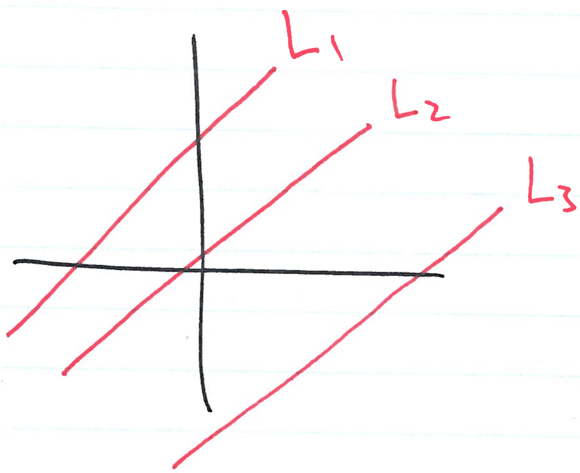
("$||$" means parallel) Symmetric: "For any two lines L_1, L_2,

if $L_1 || L_2$ or $L_1 = L_2$ then $L_2 || L_1$ or $L_2 = L_1$"

This is essentially true by definition

Transitive: "If $L_1 \parallel L_2$ and $L_2 \parallel L_3$,

then $L_1 \parallel L_3$"



Suppose $L_1 \parallel L_2$. Then they have the same slope m.

slope of L_1 and of L_2 $= m \in \mathbb{R} \cup \{\infty\}$

for vertical lines, let's say they have "slope ∞"

Since $L_2 \parallel L_3$, L_3 has the same slope as L_2.

So L_3 has slope m.

Therefore, L_1 and L_3 have the same slope m.

So $L_1 \parallel L_3$. ∎

Comment: For every possible slope $m \in \mathbb{R} \cup \{\infty\}$, the set $\{\text{Lines of slope } m\}$ is an equivalence class.

Exercise 1: Reflexive: For every real number a, $a-a=0$ and $0 \in \mathbb{Z}$.
Therefore aRa.

Symmetric: Suppose a,b are real numbers and aRb.
Then $a-b \in \mathbb{Z}$. Write $a-b=n$.
Then $b-a=-n$, which is also an integer. (The negation of an integer is an integer)
So bRa.

Transitive: Suppose a,b,c are real numbers s.t. aRb and bRc.
Then $a-b \in \mathbb{Z}$ Write $a-b=m$
and $b-c \in \mathbb{Z}$ $b-c=n$.
Then $a-c=(a-b)+(b-c)=m+n$

which is an integer (The sum of two integers is an integer)

So aRc. ■

Exercise 2: Reflexive: For every ~~real number~~ integer a, $3a+5a=8a=2\cdot(4a)$
which is even.

Symmetric: Suppose $a,b \in \mathbb{Z}$ and $3a+5b$ is even (ie aRb)
Then $3a+5b=2x$ for some $x \in \mathbb{Z}$.
Then
$$\begin{aligned} 3b+5a &= 5b+3a-2b+2a \\ &= 2x-2b+2a \\ &= 2(x-b+a) \end{aligned}$$
so $3b+5a$ is even. So bRa.

Transitive: Suppose $a,b,c \in \mathbb{Z}$ and $3a+5b$ is even and $3b+5c$ is even.
Then $3a+5c=(3a+5b)+(3b+5c)-8b$.
Let $3a+5b=2x$ and $3b+5c=2y$ for some $x,y \in \mathbb{Z}$.
$$\begin{aligned} 3a+5c &= 2x+2y-8b \\ &= 2(x+y-4b) \end{aligned}$$
is even. So aRc. ■